

Supersymmetric Quantum Mechanics for String-Bits^{*}

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ABSTRACT

We develop possible versions of supersymmetric single particle quantum mechanics, with application to superstring-bit models in view. We focus principally on space dimensions $d = 1, 2, 4, 8$, the transverse dimensionalities of superstring in 3, 4, 6, 10 space-time dimensions. These are the cases for which “classical” superstring makes sense, and also the values of d for which Hooke’s force law is compatible with the simplest superparticle dynamics. The basic question we address is: When is it possible to replace such harmonic force laws with more general ones, including forces which vanish at large distances? This is an important question because forces between string-bits that do not fall off with distance will almost certainly destroy cluster decomposition. We show that the answer is affirmative for $d = 1, 2$, negative for $d = 8$, and so far inconclusive for $d = 4$.

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1 Introduction

In string-bit models[1, 2], string is viewed as a polymer molecule, a bound system of point-like constituents which enjoy a Galilei invariant dynamics. This can be consistent with Poincaré invariant string, because the Galilei invariance of string-bit dynamics is precisely that of the transverse space of light-cone quantization. If the string-bit description of string is correct, ordinary nonrelativistic many-body quantum mechanics is the appropriate framework for string dynamics. Of course, for superstring-bits, this quantum mechanics must be made supersymmetric[3, 4].

One virtue of the string-bit idea is that the forces which bind the bits into string are also responsible for the interaction between different parts of string. However, this double duty poses a difficult challenge. It is no longer sufficient to posit non-interacting “free” string as the fundamental degree of freedom upon which interactions are imposed, for example, by adding non quadratic terms to a “string-field” Lagrangian. In order to be able to define a scattering matrix, the composite polymers must obey cluster decomposition: the interaction between widely separated polymers must vanish sufficiently rapidly. Thus the simple expedient of modelling the bond between nearest neighbors on a polymer by a harmonic “spring” force (Hooke’s Law) will not do. Such a long-range force would be indirectly felt between constituents in distant polymers, violating clustering. The only forces in the fundamental string-bit Hamiltonian should be of short range.

Supersymmetric quantum mechanics in one spatial dimension is well known to allow an essentially arbitrary potential of the form[5]

$$V(x) = W^2(x) \pm W'(x), \quad (1.1)$$

where W is the so-called superpotential. The extension of supersymmetric quantum mechanics to higher dimensions and to situations involving more than the minimal number of supercharges has been explored by various authors[6, 7, 8]. These works address the problem of constructing superalgebras with varying numbers of supercharges, and they conclude that the possibilities are strikingly limited. For example, within their working assumptions the only non-gauge models with more than four supercharges require harmonic potentials. An exception occurs when there is a gauge invariance (that is when there are first class constraints) and the superalgebra is allowed to close *modulo* a term which vanishes on gauge invariant states. In the latter case the claim is that one is limited to those quantum mechanical models that result from dimensionally reducing a supersymmetric Yang-Mills field theory[6, 7]. Our work does not conflict with any of these conclusions, but we relax the assumption that the supercharges be linear in the fermionic dynamical variables. At the same time we shall narrow the field of possibilities by requiring rotational invariance, which is not demanded in the articles cited above.

The reason we want rotational invariance is that we regard the one particle quantum mechanics we explore in this paper as a sub-sector of a many body Galilei invariant quantum mechanics, *e.g.* the center of mass dynamics of a two particle system. For the same reason, we retain another restrictive assumption of [6, 7, 8], that *the supercharges be linear in the particle momentum*. This will then naturally implement the requirement from Galilei invariance that each particle momentum enter the hamiltonian *quadratically*. In this context it is worthwhile pointing out that without some such restriction, the problem of exhibiting a superalgebra with a highly nonharmonic hamiltonian is trivially solved. Indeed, pick any hermitian operator Ω acting on the state space of a (non-supersymmetric) quantum particle. Adjoin to the system of quantum operators, spinor valued elements S^A of a Clifford algebra which commute with all other operators. Then $S^A\Omega$ satisfies the superalgebra with $H = \Omega^2/2$. Clearly there is very little interesting about this supersymmetric system: all we have is multiple copies of an original non-supersymmetric system, with hamiltonian

$\Omega^2/2$, whose only special feature is a positive definite hamiltonian.

As mentioned above, the larger context of this work is to find a satisfactory string-bit model for superstring in which the forces between bits fall off with distance. In earlier work[9] Bergman and I constructed such a model with full Galilei supersymmetry, but it had the fatal flaw that the potential energy of the two bit system also vanished at large distance. Since the model was supersymmetric this meant that any bound state would have zero binding energy. In fact the two-bit system did not possess even a zero energy bound state. Thus we place particular emphasis on models in which the forces vanish at large distance while the potential energy approaches a non-vanishing (positive) constant. In such models a supersymmetric ground state will automatically have nonvanishing binding energy.

We shall only consider non-gauge models in this article. Although we consider the problem of developing supersymmetric particle quantum mechanics in arbitrary dimensionalities very interesting, the fact that each dimensionality presents special features leads us to narrow our attention to the dimensionalities of interest for string theory: $d = 1, 2, 4, 8$, the transverse dimensionalities for 3, 4, 6, and 10 dimensional space-time. It is precisely for these dimensionalities that it is possible to introduce a harmonic potential into superparticle dynamics. Our main interest in this article is to inquire whether short range forces are also possible for these cases. For $d = 1$ Witten's analysis shows that they are. We shall find that only harmonic forces are possible in an 8-dimensional model with 8 supercharges but short-range forces can be introduced in a 2-dimensional model with 2 supercharges. In the latter case we shall exhibit supercharges with terms *cubic* in the fermionic variables. This explicitly demonstrates that the assumption of earlier authors that the supercharges are linear in fermionic variables is overly restrictive.

The rest of the paper is organized as follows. In Section 2 we discuss supersymmetry in models with only harmonic forces. Although such models have been studied before, we discuss them here to set notation and also to display the supercharges as *spinors* of the rotation group. In Section 3 we consider generalizations to non-harmonic forces, and find the severe restrictions mentioned above. The models we present and study certainly resemble some of those in the literature we have cited. There are, however, new features that arise in some of our models chiefly because, motivated by superstring applications, we include more than the minimal number of fermionic degrees of freedom. In Section 4 we analyze the 2-dimensional models in some detail, exhibiting cases in which there is a supersymmetric ground state with finite binding energy. Finally, we conclude in Section 5 with a discussion of the application of our results to superstring-bit models. We deal with the case $d = 4$ in an appendix.

2 Harmonic Forces

In the main text, we shall limit our constructions of supersymmetry algebras to those of hermitian supercharges Q_A where A is a $2^{d/2}$ dimensional Majorana spinor index for $O(d)$. We also allow a further restriction to a Majorana-Weyl spinor when possible. Then d is limited to those dimensions for which the Dirac gamma matrices can be chosen real and symmetric:

$$\gamma_{AB}^k = \gamma_{AB}^{k*} = \gamma_{BA}^k \quad \{\gamma^k, \gamma^l\} = 2\delta_{kl} \quad (2.2)$$

For application to superstring the interesting dimensions are $d = 2, 8^*$ (the transverse dimensionalities for string in 4 and 10 dimensional space-time, respectively). Similarly we choose Grassmann

*The other case interesting for superstring is $d = 4$ for which Majorana gamma matrices don't exist. Thus Q_A , S^A , and \tilde{S}^A are all non-hermitian. This case is also consistent with harmonic forces, provided a Weyl restriction is made. We shall discuss this case in an appendix to keep the line of argument uncluttered.

odd variables motivated by this string application. We introduce a pair of hermitian spinor valued Clifford algebra elements S^A, \tilde{S}^A , satisfying

$$\begin{aligned}\{S^A, S^B\} &= 2\delta_{AB} & \{\tilde{S}^A, \tilde{S}^B\} &= 2\delta_{AB} \\ \{S^A, \tilde{S}^B\} &= 0.\end{aligned}\tag{2.3}$$

Let K be the dimensionality of the spinors. Then $K = 2^{d/2}$ for Dirac spinors and $K = 2^{d/2-1}$ for Weyl spinors.

If the coordinates appear quadratically in the hamiltonian, they should appear linearly in the supercharges. This motivates the following *ansatz* for the supersymmetry generators

$$Q_A = \mathbf{p} \cdot \gamma_{AB} S^B + k\mathbf{x} \cdot \gamma_{AB} \tilde{S}^B.\tag{2.4}$$

By construction, the Q_A transform as the components of a spinor under rotations. A simple calculation yields

$$\begin{aligned}\{Q_A, Q_B\} &= 2\delta_{AB}(\mathbf{p}^2 + k^2\mathbf{x}^2) \\ &\quad + ik(\gamma_{AC} \cdot \gamma_{BD} + \gamma_{BC} \cdot \gamma_{AD})\tilde{S}^C S^D\end{aligned}\tag{2.5}$$

The supersymmetry algebra demands that this anticommutator be $4\delta_{AB}H$. If the r.h.s. happens to be proportional to the Kronecker delta, the coefficient will *define* the hamiltonian H . To achieve this, we clearly need the combination of gamma matrices to satisfy a special identity:

$$\gamma_{AC} \cdot \gamma_{BD} + \gamma_{BC} \cdot \gamma_{AD} = 2\delta_{AB}L_{CD}\tag{2.6}$$

for some matrix L . In fact, simultaneous validity of Eq. (2.2) and Eq. (2.6) implies that $L_{CD} = \delta_{CD}$ and $K = d$. Clearly such an identity will not hold generally, but by using the Fierz identities one can test whether it holds in various dimensions. We shall examine it for $d = 2, 8$, the only cases for which $K = d$ and the Majorana representation is possible.

The Fierz identities use a complete set of spinor matrices to interchange indices in the outer product of two matrices. The canonical basis of spinor matrices is taken to be the anti-symmetrized products of gamma matrices:

$$\gamma^{k_1 k_2 \dots k_n} \equiv \gamma^{[k_1} \gamma^{k_2} \dots \gamma^{k_n]}\tag{2.7}$$

where the square brackets denote the anti-symmetrized sum over all permutations of the n indices, normalized by dividing by $n!$. For d dimensions $n = 0, 1, 2, \dots, d$, with the case $n = 0$ understood as the identity matrix. Then the Fierz identity we need takes the form

$$\gamma_{AC} \cdot \gamma_{BD} = \sum_{n=0}^d C_n \gamma_{AB}^{k_1 k_2 \dots k_n} \gamma_{CD}^{k_1 k_2 \dots k_n}\tag{2.8}$$

Since the supersymmetry involves the l.h.s. symmetrized in A, B , the desired result is obtained when the r.h.s. involves only the identity and antisymmetric matrices. One easily finds

$$\gamma^{k_1 k_2 \dots k_n T} = (-)^{n(n-1)/2} \gamma^{k_1 k_2 \dots k_n}\tag{2.9}$$

so the antisymmetric ones are $n = 2, 6, 10, \dots$ and $n = 3, 7, 11, \dots$. Also C_n is proportional to $d - 2n$, so the term with $n = d/2$ is never present. Thus for $d = 2$ only $n = 0, 2$ appear in the Fierz identity and since the latter is antisymmetric, the supersymmetry algebra closes by default. The required identity Eq. (2.6) assumes the form

$$d = 2 : \quad \gamma_{AC} \cdot \gamma_{BD} + \gamma_{BC} \cdot \gamma_{AD} = 2\delta_{AB}\delta_{CD}\tag{2.10}$$

For $d = 8$ the situation is a bit more complex: all $n \neq 4$ appear in the Fierz identity. Of these $n = 0, 1, 5, 8$ are symmetric and contribute to the anticommutator of supercharges. As is well known from the Green-Schwarz formulation of the light-cone superstring, or from the triality symmetry of $SO(8)$, the way to close the algebra, is to make a Weyl restriction: with the index of Q_A restricted to a subset on which the chirality matrix $\gamma_9 \equiv \gamma^1 \gamma^2 \cdots \gamma^8$ is proportional to the identity. Then the $n = 1, 5$ terms won't enter the anticommutator (since they connect indices with opposite values of γ_9), and the $n = 8$ term will simply double the $n = 0$ term. When the Weyl restriction is made, the supercharges Q will have chirality opposite to that of the spinors S . The customary dotted index notation is then useful: the subset of spinor indices A with chirality $+1$ is denoted a and the subset with chirality -1 is denoted \dot{a} . For definiteness we shall take the spinors S to have undotted indices. Then the identity Eq. (2.6) takes the form

$$d = 8 : \quad \gamma_{\dot{a}c} \cdot \gamma_{bd} + \gamma_{bc} \cdot \gamma_{\dot{a}d} = 2\delta_{\dot{a}b}\delta_{cd} \quad (2.11)$$

In summary, we have reviewed the extension of a quantum mechanical harmonic oscillator to a supersymmetric system for $d = 2, 8$. In doing so we have limited our discussion to the Grassmann degrees of freedom suggested by superstring and have assumed a minimal set of *hermitian* spinor valued supercharges and the *ansatz* Eq. (2.4). A similar *ansatz* works for the case $d = 4$ with non-hermitian supercharges. In fact the way supersymmetry is realized in this construction is essentially identical to the way it is realized on a single mode of superstring. The supersymmetry algebra does not close for other dimensionalities if Eq. (2.4) is assumed. We do not know to what extent constructions for other dimensionalities might be made to work if, for example, the supercharges are allowed to depend on the spinor variables in a nonlinear way. In the following section we shall find such nonlinearities are inevitable with non-harmonic forces, but for harmonic forces, we have chosen not to investigate them.

3 Non-Harmonic Forces

Although harmonic forces can provide useful models in certain physical situations, their extreme long range character threatens disastrous violations of cluster decomposition. Thus the forces among string-bits should, at the very least, vanish for large separations. In this section we examine the possibilities for generalizing the supersymmetric quantum mechanics models of the previous section to ones with non-harmonic forces, including those of short range. Witten's one dimensional examples show that there is no logical barrier to such a generalization. However, for $d > 1$ there are highly non-trivial constraints that must be satisfied. Our efforts will only be fully successful for $d = 2$.

In light of the trivial realization of supersymmetry mentioned in the introduction, to get dynamically interesting models we must demand more than an algebraic realization of supersymmetry. For example in supersymmetric field theory, locality provides a powerful additional restriction on the dynamics. Locality is not really applicable to one particle quantum mechanics. But an analogously powerful restriction is provided by requiring the particle momentum to enter the hamiltonian quadratically. Thus it is reasonable to require that the momentum dependence of the supercharges in the presence of non-harmonic forces be identical to that of the free or harmonic case:

$$Q_A = \mathbf{p} \cdot \gamma_{AB} S^B + \hat{Q}^A(\mathbf{x}, S, \tilde{S}). \quad (3.1)$$

Requiring supersymmetry on this *ansatz* does indeed narrow the possibilities drastically, as we shall see.

To begin write out the anticommutator,

$$\begin{aligned}\{Q_A, Q_B\} &= 2\mathbf{p}^2\delta_{AB} + \{\hat{Q}_A, \hat{Q}_B\} \\ &+ \frac{1}{2}(\gamma_{AC}^k\{p^k, \{S^C, \hat{Q}_B\}\} + \gamma_{BC}^k\{p^k, \{S^C, \hat{Q}_A\}\}) \\ &+ \frac{i}{2}(\gamma_{AC} \cdot \nabla[\hat{Q}_B, S^C] + \gamma_{BC} \cdot \nabla[\hat{Q}_A, S^C]),\end{aligned}\tag{3.2}$$

and impose that each power of p^k be separately proportional to a δ_{AB} . There are no constraints from the quadratic terms, but the linear terms give

$$\gamma_{AC}^k\{S^C, \hat{Q}_B\} + \gamma_{BC}^k\{S^C, \hat{Q}_A\} = 2\Omega^k\delta_{AB},\tag{3.3}$$

while the momentum independent terms give

$$\begin{aligned}\{\hat{Q}_A, \hat{Q}_B\} &+ \frac{i}{2}(\gamma_{AC} \cdot \nabla[\hat{Q}_B, S^C] + \gamma_{BC} \cdot \nabla[\hat{Q}_A, S^C]) \\ &= 4V\delta_{AB}\end{aligned}\tag{3.4}$$

If these constraints can be satisfied the implied supersymmetric hamiltonian would be

$$H = \frac{\mathbf{p}^2}{2} + \frac{1}{4}(p^k\Omega^k + \Omega^k p^k) + V.\tag{3.5}$$

We shall draw out the consequences of these constraints in stages. First develop \hat{Q}_A in an expansion in antisymmetrized monomials of S^A :

$$\hat{Q}_A = \sum_{k=0}^K M_A^{B_1 \dots B_k}(\mathbf{x}, \tilde{S}) S^{[B_1} S^{B_2} \dots S^{B_k]}.\tag{3.6}$$

Of course it is understood that M is Grassmann odd (even) if k is even (odd). Without loss of generality we can take M to be completely antisymmetric in its upper indices. The upper limit K on the sum will be the spinor dimensionality, $2^{d/2}$ for Majorana-Dirac spinors and $2^{(d-2)/2}$ for Majorana-Weyl spinors. For the cases of particular interest to us ($d = 2$ Dirac, $d = 8$ Weyl) the upper limit is numerically equal to d . Applying the constraints Eq. (3.3) leads to no restriction on the the $k = 0$ case and for $k > 0$ amounts to:

$$\gamma_{AC}^i M_B^{CB_2 \dots B_k} + \gamma_{BC}^i M_A^{CB_2 \dots B_k} = 2\delta_{AB} A_{B_2 \dots B_k}^i,\tag{3.7}$$

where we have explicitly used the antisymmetry of M in all its upper indices and A is as yet undetermined. When Eq. (2.10) or Eq. (2.11) hold (for us this requires $d = 2, 8$), it is easy to solve Eq. (3.7) using them and the identities Eq. (2.2): simply put $B = A$ to get

$$\gamma_{AC}^i M_A^{CB_2 \dots B_k} = A_{B_2 \dots B_k}^i.\tag{3.8}$$

Then Eq. (2.10) with $A = B$ reads $\gamma_{AC} \cdot \gamma_{AD} = \delta_{CD}$, so that

$$M_A^{CB_2 \dots B_k} = \gamma_{AC}^i A_{B_2 \dots B_k}^i.\tag{3.9}$$

But for $k > 1$ this factorized form is inconsistent with the antisymmetry of M in its upper indices: the r.h.s. must vanish for $C = B_2$ which would imply $A_{B_2 \dots B_k}^i = 0$. This follows from the Clifford

algebra Eq. (2.2). Thus only the terms in Eq. (3.6) with $k = 0, 1$ contribute so that the supercharges simplify to

$$Q_A = (\mathbf{p} + \mathbf{A}(\mathbf{x}, \tilde{S})) \cdot \gamma_{AB} S^B + M_A(\mathbf{x}, \tilde{S}). \quad (3.10)$$

Notice how \mathbf{A} enters exactly as the vector potential of a gauge field. The simplification so far is not so surprising: we have begun with the restriction that Q is linear in the components of \mathbf{p} and have found that the dependence on S , the spinor naturally associated with the momentum, must also be linear. Although the simple form Eq. (3.10) has only been proved to be forced when $d = 2, 8$, we shall analyze it as an *ansatz* for generic d in the following.

With the form of supercharges in Eq. (3.10), the supersymmetry algebra closes up to terms independent of momentum. Requiring complete closure will put constraints on \mathbf{A} and M_A . Again it is efficient to organize the terms that arise from the anticommutator according to powers of the spinor S^B :

$$\begin{aligned} \{Q_A, Q_B\} &= 2(\mathbf{p} + \mathbf{A})^2 \delta_{AB} + \{M_A, M_B\} \\ &+ ([M_B, (\mathbf{p} + \mathbf{A}) \cdot \gamma_{AC}] + [M_A, (\mathbf{p} + \mathbf{A}) \cdot \gamma_{BC}]) S^C \\ &+ [p^k + A^k, p^m + A^m] \gamma_{AC}^k \gamma_{BD}^m S^{[C} S^{D]}. \end{aligned} \quad (3.11)$$

Each power of S must be separately proportional to δ_{AB} . Look first at the term quadratic in S . The Dirac matrices enter in the combination

$$\gamma_{AC}^k \gamma_{BD}^m - \gamma_{AD}^k \gamma_{BC}^m - \gamma_{AC}^m \gamma_{BD}^k + \gamma_{AD}^m \gamma_{BC}^k. \quad (3.12)$$

In general dimensionality this combination will *not* be proportional to δ_{AB} , in which case closure of the supersymmetry algebra will impose linear relations among the components of $[p^k + A^k, p^m + A^m] \equiv iF^{mk}$. In sufficiently high dimensionality, there will be so many independent conditions to force the vanishing of all components:

$$\begin{aligned} [p^k + A^k, p^m + A^m] &= i(\partial^m A^k - \partial^k A^m) - [A^m, A^k] \\ &\equiv iF^{mk} = 0, \quad \text{generic } d. \end{aligned} \quad (3.13)$$

This constraint on F^{mk} might be relaxed, partially or completely, in specific dimensionalities. In view of rotational invariance a partial relaxation is an option only in $d = 4$, where self-duality can provide a rotationally invariant linear relation.

The generic constraints Eq. (3.13) are so powerful, that it is worth pursuing their consequences once and for all. Thinking of F as a nonabelian field strength shows us immediately that the solution of this constraint is that \mathbf{A} is a “pure gauge”

$$\mathbf{A} = \Omega^\dagger i \nabla \Omega, \quad \Omega^\dagger \Omega = I, \quad \text{generic } d. \quad (3.14)$$

But with \mathbf{A} of this form, the supercharges are unitary equivalents of charges with $\mathbf{A} = 0$. Thus without loss of generality we can take

$$Q_A = \mathbf{p} \cdot \gamma_{AB} S^B + M_A(\mathbf{x}, \tilde{S}), \quad \text{generic } d. \quad (3.15)$$

With closure conditions

$$\{M_A, M_B\} = 4V \delta_{AB} \quad (3.16)$$

$$(\nabla M_B \cdot \gamma_{AC} + \nabla M_A \cdot \gamma_{BC}) = 2\Psi_C \delta_{AB}. \quad (3.17)$$

The second closure condition Eq. (3.17) can be inverted by setting $B = A$, multiplying both sides by γ_{AC} , summing over C , and using the Clifford algebra:

$$\nabla M_A = \gamma_{AC} \Psi_C. \quad (3.18)$$

Note that this form satisfies Eq. (3.17) only if Eq. (2.10) or Eq. (2.11) hold, *i.e.* only if $d = 2, 8$. The integrability condition for the last displayed equation is

$$(\nabla^i \gamma^j - \nabla^j \gamma^i) \Psi = 0, \quad (3.19)$$

where matrix multiplication is understood. For fixed *distinct* i, j , multiply this equation on the left by the matrix $\gamma^i \gamma^j = -\gamma^j \gamma^i$; this leads to

$$(\nabla^i \gamma^i + \nabla^j \gamma^j) \Psi = 0 \quad \text{each distinct pair } i, j. \quad (3.20)$$

For $d > 2$ this in turn implies that $\nabla_i \gamma_i \Psi_C = 0$ for each i , and, since γ^i is an invertible matrix, Ψ_C is *independent* of the coordinates \mathbf{x} . So we conclude that Eq. (3.17) holds for $d > 2$ if and only if

$$M_A = M_A^0(\tilde{S}) + \mathbf{x} \cdot \gamma_{AC} \Psi_C(\tilde{S}). \quad (3.21)$$

For $d = 2$ the integrability condition is less stringent, implying only that $\Psi \equiv \Psi_1 - i\Psi_2$ is a holomorphic function of $z = x_1 + ix_2$. But rotational invariance (see the next section), *i.e.* that Ψ_C transform as a spinor with spin $\pm 1/2$, is enough to force the linear dependence on coordinates Eq. (3.21) for $d = 2$ as well. So we are nearly back to the harmonic force case of the previous section. It only remains to impose the other closure condition Eq. (3.16). Look first at the term in the anticommutator quadratic in \mathbf{x} , which yields

$$(\gamma_{AC}^m \gamma_{BD}^n + \gamma_{AC}^n \gamma_{BD}^m) \{\Psi_C, \Psi_D\} = 4\delta_{AB} V^{mn}. \quad (3.22)$$

by setting $B = A$, summing over A , and using the Clifford algebra, we easily see that $V^{mn} \propto \delta_{mn}$ so put $V^{mn} = V_2 \delta_{mn}$. Then setting $B = A$ (but not summing) and inverting in the now familiar way, we find

$$\{\Psi_C, \Psi_D\} = 2V_2 \delta_{mn} \gamma_{AC}^m \gamma_{AD}^n = 2V_2 \delta_{CD}. \quad (3.23)$$

It follows that V_2 commutes with Ψ_C . Thus $\Psi_C / \sqrt{V_2}$ which is a function only of \tilde{S} is a spinor whose components satisfy a Clifford algebra isomorphic to that \tilde{S}_C . Hence it is unitarily equivalent to the latter and $V_2 \equiv k^2$ is a positive c -number. Thus we can identify Ψ_C with $k\tilde{S}$ and write

$$M_A = M_A^0(\tilde{S}) + k\mathbf{x} \cdot \gamma_{AC} \tilde{S}_C. \quad (3.24)$$

But now an identical argument to that which led from Eq. (3.1) to Eq. (3.10) shows that $M_A^0 \propto \mathbf{v} \cdot \gamma_{AC} \tilde{S}^C$, where \mathbf{v} is a c -number vector. Since we are assuming rotational invariance this vector must vanish. We then conclude that for generic dimensionality, the *ansatz* of Eq. (3.10), which is a logical consequence of Eq. (3.1) for $d = 2, 8$, leads inevitably to the supersymmetric harmonic oscillator discussed in the previous section.

There remains the loophole that in certain specific dimensionalities the combination of Dirac gamma matrices Eq. (3.12) might fortuitously be proportional to δ_{AB} , or it might not have enough independent components to force *all* components of F to vanish. Here we only consider the cases $d = 2, 8$. The case of $d = 4$ where self-duality is a possibility is treated in the appendix. First develop a Fierz-like expansion of Eq. (3.12):

$$\begin{aligned} \gamma_{AC}^k \gamma_{BD}^m - \gamma_{AD}^k \gamma_{BC}^m - \gamma_{AC}^m \gamma_{BD}^k + \gamma_{AD}^m \gamma_{BC}^k \\ = \sum_{n=0}^d C_{CDkm}^{k_1 k_2 \dots k_n} \gamma_{AB}^{k_1 k_2 \dots k_n}. \end{aligned} \quad (3.25)$$

Because the l.h.s. is symmetric in AB , only the terms with $n = 0, 4, 8, \dots$ and $n = 1, 5, \dots$ will enter the sum on the r.h.s.

For $d = 8$ we have made the Weyl restriction and have agreed to take $AB \rightarrow \dot{a}\dot{b}$ to be dotted and $CD \rightarrow cd$ to be undotted. Then the terms with n odd will not appear and those with $n > 4$ give nothing new. Thus we are limited to the terms with $n = 0, 4$. Moreover, since the l.h.s. is antisymmetric in cd the coefficient $C_{cdkm}^{k_1 k_2 \dots k_n}$ must be a linear combination of the matrices $\gamma_{cd}^{l_1 l_2}$. By rotational invariance, the expansion simplifies to

$$\begin{aligned} \gamma_{\dot{a}c}^k \gamma_{bd}^m - \gamma_{\dot{a}d}^k \gamma_{bc}^m - \gamma_{\dot{a}c}^m \gamma_{bd}^k + \gamma_{\dot{a}d}^m \gamma_{bc}^k \\ = C_0 \delta_{\dot{a}\dot{b}} \gamma_{cd}^{km} + C_4 \gamma_{\dot{a}\dot{b}}^{km k_1 k_2} \gamma_{cd}^{k_1 k_2} \end{aligned} \quad d = 8. \quad (3.26)$$

By tracing the indices $\dot{a}\dot{b}$, one easily finds that $C_0 = 1/2$. Then multiplying by $\gamma_{dc}^{r_1 r_2}$ and summing over c, d allows one to conclude that $C_4 = 1/4 \neq 0$. There are 35 independent components of $\gamma_{\dot{a}\dot{b}}^{klmn}$, which is more than enough to force all 28 components of F to vanish. Specifically, supersymmetry requires

$$F^{km} \gamma_{\dot{a}\dot{b}}^{km k_1 k_2} \gamma_{cd}^{k_1 k_2} = 0 \quad (3.27)$$

for all c, d, \dot{a}, \dot{b} . Multiplying the l.h.s. by $\gamma_{\dot{b}\dot{a}}^{l_1 l_2 l_3 l_4} \gamma_{dc}^{m_1 m_2}$, and summing over the repeated subscripts, reexpresses this condition as

$$F^{km} (\delta_{[l_1}^k \delta_{l_2}^m \delta_{l_3}^{k_1} \delta_{l_4}^{k_2}] + \epsilon^{km k_1 k_2 l_1 l_2 l_3 l_4}) = 0. \quad (3.28)$$

Choosing $k_1, k_2, l_1, l_2, l_3, l_4$ all different singles out a unique component of F , so for $d = 8$ we conclude that, within the *ansatz* Eq. (3.1), only harmonic forces are consistent with the supersymmetry algebra.

For $d = 2$, there is no Weyl restriction, and both $n = 0, 1$ could appear in the expansion. However, the antisymmetry in CD means that the CD dependence must be carried by $\gamma_{CD}^{l_1 l_2} = \epsilon_{l_1 l_2} (\gamma^1 \gamma^2)_{CD}$. Thus rotational invariance excludes the presence of $n = 1$. We find

$$\begin{aligned} \gamma_{AC}^k \gamma_{BD}^m - \gamma_{AD}^k \gamma_{BC}^m - \gamma_{AC}^m \gamma_{BD}^k + \gamma_{AD}^m \gamma_{BC}^k \\ = 2\delta_{AB} \epsilon^{km} (\gamma^1 \gamma^2)_{CD} \end{aligned} \quad d = 2. \quad (3.29)$$

Thus \mathbf{A} is not constrained to be a pure gauge, and the closure condition Eq. (3.17) is relaxed to

$$\begin{aligned} (\nabla M_B - i[M_B, \mathbf{A}]) \cdot \gamma_{AC} + (\nabla M_A - i[M_A, \mathbf{A}]) \cdot \gamma_{BC} \\ = 2\Psi_C \delta_{AB} \end{aligned} \quad (3.30)$$

In the next section we shall find that these relaxed constraints leave room for an essentially arbitrary rotationally invariant potential.

4 Supersymmetric Quantum Mechanics in 2 Dimensions.

In two dimensions, the two component spinor \tilde{S} enters M at most linearly and can enter \mathbf{A} at most quadratically. This makes a general analysis of this case mercifully tractable. Thus M_A can be assumed to be, putting $\Gamma \equiv \gamma^1 \gamma^2$, and using rotational invariance,

$$\begin{aligned} M_A = & [(B_1(\mathbf{x}^2) + B_2(\mathbf{x}^2)\Gamma)\mathbf{x} \cdot \gamma]_{AC} \tilde{S}^C \\ & + (C_1(\mathbf{x}^2) + C_2(\mathbf{x}^2)\Gamma)_{AC} \tilde{S}^C \end{aligned} \quad (4.1)$$

where the coefficients are all real. A short evaluation using the closure condition Eq. (3.16) then shows that either $C_1 = C_2 = 0$ or $B_1 = B_2 = 0$. Furthermore a unitary transformation of the form $\exp(\alpha(\mathbf{x}^2)\Gamma_{CD}\tilde{S}^C\tilde{S}^D/4)$ can be used to rotate B_2 (or C_2) away, so we can assume

$$M_A = B(\mathbf{x}^2)\mathbf{x} \cdot \gamma_{AC}\tilde{S}^C \quad \text{or} \quad M_A = C(\mathbf{x}^2)\tilde{S}^A \quad (4.2)$$

Similarly, rotational invariance restricts \mathbf{A} to the form

$$\mathbf{A} = \mathbf{A}_1(\mathbf{x}) + \frac{1}{4}\mathbf{A}_2(\mathbf{x})i\Gamma_{CD}\tilde{S}^C\tilde{S}^D. \quad (4.3)$$

It follows that

$$-i[\tilde{S}^A, \mathbf{A}] = \mathbf{A}_2\Gamma_{AC}\tilde{S}^C, \quad (4.4)$$

Then in the case that $C(\mathbf{x}^2)$ is nonzero it is immediate that Eq. (3.30) cannot be satisfied unless C is a constant and $\mathbf{A}_2 = 0$. The other case with $B \neq 0$ involves

$$\begin{aligned} \nabla M_B - i[M_B, \mathbf{A}] &= (B(\mathbf{x}^2)\gamma_{BC} + 2\mathbf{x}B'(\mathbf{x}^2)\mathbf{x} \cdot \gamma_{BC} \\ &\quad + B(\mathbf{x}^2)\mathbf{A}_2\mathbf{x} \cdot (\gamma\Gamma)_{BC})\tilde{S}^C. \end{aligned} \quad (4.5)$$

This result allows a nontrivial solution of Eq. (3.30) because of the special properties of two dimensions. The dual of a vector $v_D^i \equiv \epsilon^{ij}v^j$ is a vector. Furthermore $\gamma\Gamma = \gamma_D$ from which follows $\mathbf{x} \cdot \gamma\Gamma = -\mathbf{x}_D \cdot \gamma$. Finally we have the identity

$$v_D^i v_D^j = \epsilon^{ik}v^k \epsilon^{jl}v^l = \delta_{ij}\mathbf{v}^2 - v^i v^j. \quad (4.6)$$

Applying these special properties to the first term on the l.h.s of Eq. (3.30), leads to (with $\mathbf{A}_2 = -2\mathbf{x}_D B'/B$)

$$\begin{aligned} (\nabla M_B - i[M_B, \mathbf{A}]) \cdot \gamma_{AC} &= (B(\mathbf{x}^2)\gamma_{BD} \cdot \gamma_{AC} + \\ &\quad 2B'(\mathbf{x}^2)\mathbf{x} \cdot \gamma_{AC}\mathbf{x} \cdot \gamma_{BD} - B(\mathbf{x}^2)\mathbf{A}_2 \cdot \gamma_{AC}\mathbf{x}_D \cdot \gamma_{BD})\tilde{S}^D \\ &= (B(\mathbf{x}^2) + 2\mathbf{x}^2 B'(\mathbf{x}^2))\gamma_{BD} \cdot \gamma_{AC}\tilde{S}^D, \end{aligned} \quad (4.7)$$

Doing the same to the second term and making use of Eq. (2.10) shows that Eq. (3.30) holds with

$$\Psi_C = (B(\mathbf{x}^2) + 2\mathbf{x}^2 B'(\mathbf{x}^2))\tilde{S}^C. \quad (4.8)$$

Putting all the pieces together, we conclude that for $d = 2$ the supersymmetry generators can be taken to be

$$\begin{aligned} Q_A &= \left(\mathbf{p} + \mathbf{A}_1(\mathbf{x}) - \mathbf{x}_D \frac{B'(\mathbf{x}^2)}{2B(\mathbf{x}^2)} i\Gamma_{CD}\tilde{S}^C\tilde{S}^D \right) \cdot \gamma_{AB}S^B \\ &\quad + B(\mathbf{x}^2)\mathbf{x} \cdot \gamma_{AC}\tilde{S}^C. \end{aligned} \quad (4.9)$$

More general forms are unitarily equivalent to this. Incidentally, it is apparent from Eq. (4.9) that the case $\mathbf{A} = 0$ does indeed reduce to the supersymmetric oscillator, as we mentioned in the previous section.

Before we analyze some of these two dimensional models, it is worthwhile noting a particularly simple way of understanding the structure of Eq. (4.9). Since Q_A has two hermitian components, we can combine them into a single non-hermitian supercharge, with superalgebra:

$$\begin{aligned} Q &\equiv \frac{1}{\sqrt{2}}(Q_1 + iQ_2) \quad Q^2 = 0 \\ \{Q, Q^\dagger\} &= Q_1^2 + Q_2^2 = 4H. \end{aligned} \quad (4.10)$$

The last equation may be taken simply as the *definition* of the hamiltonian H . Thus the only nontrivial content of supersymmetry is the nilpotency of Q . If one nilpotent Q_0 can be found (for example, the supercharge for harmonic forces) then $Q = YQ_0Y^{-1}$, where Y is *any* invertible operator, will also be nilpotent. Of course if Y is unitary, the dynamics is equivalent to that of the original system, and nothing new is obtained. However, if $Y^\dagger \neq Y^{-1}$, one obtains by this device a completely new system. It is easy to show that the form Eq. (4.9) is obtained in this way if Y is restricted to be a function of \mathbf{x} and \tilde{S} . That restriction is precisely what is needed to implement our requirement that the momentum dependent part of the supercharges be that of the harmonic or free system.

To facilitate the solution of these two dimensional models, it is useful to introduce a representation for S, \tilde{S} in terms of 4×4 matrices:

$$\begin{aligned} S^1 &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} & S^2 &= \begin{pmatrix} 0 & i\sigma^3 \\ -i\sigma^3 & 0 \end{pmatrix} \\ \tilde{S}^1 &= \begin{pmatrix} 0 & i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix} & \tilde{S}^2 &= \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \end{aligned} \quad (4.11)$$

At the same time let's fix the representation of the 2×2 gamma matrices as

$$\begin{aligned} \gamma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \gamma^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \Gamma &\equiv \gamma^1 \gamma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned} \quad (4.12)$$

It is also useful to identify the generator of rotations in the plane in order to keep track of the quantum numbers of the energy eigenstates. We find that the angular momentum is given by

$$J = x^1 p^2 - x^2 p^1 + \frac{i}{4}(S^1 S^2 + \tilde{S}^1 \tilde{S}^2) \quad (4.13)$$

In terms of the matrix representation Eq. (4.11) the angular momentum takes the forms

$$J = x^1 p^2 - x^2 p^1 - \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \quad (4.14)$$

from which we see that the top two components of the four component wave function have spin 0 while the bottom two components have spin $\mp 1/2$.

The matrix representations of the supercharges are

$$Q_A = \begin{pmatrix} 0 & q_A^\dagger \\ q_A & 0 \end{pmatrix} \quad (4.15)$$

where

$$\begin{aligned} q_1 &= p^2 - ip^1 \sigma^3 + \frac{B'}{B}(x^1 \sigma^3 + ix^2 I) - iB(x^2 \sigma^1 + x^1 \sigma^2) \\ q_2 &= p^1 + ip^2 \sigma^3 - \frac{B'}{B}(x^2 \sigma^3 - ix^1 I) - iB(x^1 \sigma^1 - x^2 \sigma^2) \\ &= i\sigma^3 q_1 \end{aligned} \quad (4.16)$$

One fundamental question to ask of any supersymmetric system is whether the ground state is supersymmetric, *i.e.* whether it is annihilated by the Q_A . Any such state is automatically a

zero energy eigenstate of the Hamiltonian. For the case of harmonic forces the ground state is supersymmetric and has spin zero, so it is natural to look for a supersymmetric state in the spin zero sector, so we assume $\Psi = (\psi_1, \psi_2, 0, 0)$. Denoting $\psi \equiv (\psi_1, \psi_2)$, we search for solutions of

$$\begin{aligned} q_1 \psi = [p^2 - ip^1 \sigma^3 &+ \frac{B'}{B}(x^1 \sigma^3 + ix^2 I) \\ &- iB(x^2 \sigma^1 + x^1 \sigma^2)] \psi = 0. \end{aligned} \quad (4.17)$$

Any such solution will automatically be annihilated by q_2 . In terms of components, this equation becomes the pair

$$\begin{aligned} [p^2 - ip^1 + \frac{B'}{B}(x^1 + ix^2)] \psi_1 - B[x^1 + ix^2] \psi_2 &= 0 \\ [p^2 + ip^1 - \frac{B'}{B}(x^1 - ix^2)] \psi_2 + B[x^1 - ix^2] \psi_1 &= 0. \end{aligned} \quad (4.18)$$

These equations can be directly integrated in the case of zero angular momentum, in which case we can assume that $\psi_{1,2}$ are functions of \mathbf{x}^2 , which implies $(p^2 \mp ip^1) \psi_{1,2} = \mp 2(x^1 \pm ix^2) \psi'_{1,2}$, where the prime indicates differentiation with respect to $u \equiv r^2$. Then the equations reduce to

$$\begin{aligned} -2\psi'_1 + \frac{B'}{B} \psi_1 - B\psi_2 &= 0 \\ 2\psi'_2 - \frac{B'}{B} \psi_2 + B\psi_1 &= 0. \end{aligned} \quad (4.19)$$

Taking sums and differences leads to two decoupled equations for $\psi_{\pm} = \psi_1 \pm \psi_2$, with solutions

$$\psi_{\pm}(\mathbf{x}^2) = K_{\pm} \sqrt{\frac{B(\mathbf{x}^2)}{B(0)}} \exp \left\{ \mp \int_0^{\mathbf{x}^2} du B(u) \right\}. \quad (4.20)$$

Clearly one or the other of these wave functions is normalizable provided the integral in the exponent diverges as $|\mathbf{x}| \rightarrow \infty$ sufficiently rapidly and the sign \mp is chosen according to whether the integral blows up positively or negatively. For a finite non-zero binding energy, the wave function should fall off exponentially with distance, which would require that $B(u) \sim 1/\sqrt{u}$ as $u \rightarrow +\infty$. The harmonic case corresponds to $B(u) = \text{constant}$ and gaussian wave functions.

5 Applications to String-bit Models and Concluding Remarks

We have managed to construct a supersymmetric one particle quantum mechanics with a short-range potential (*i.e.* forces vanishing at large distances) in 2 dimensions, but not in 8 dimensions. We hope this provides a useful step toward a physically satisfactory string-bit model of superstring. In an earlier work[3] Bergman and I constructed a bit model of the free type IIB superstring based on a harmonic nearest neighbor bond potential. The model possessed full Galilei supersymmetry for both $d = 2$ and $d = 8$. We can use the results of the preceding sections to relax the restriction to harmonic forces in the $d = 2$ case.

In the “bare polymer” approximation ($N_c \rightarrow \infty$) the string bit supercharges Ref.[3], acting on a single polymer state with M bits, took the form:

$$Q^A = \sum_{k=1}^M [\mathbf{p}_k \cdot \gamma_{AB} S_k^B + T_0(\mathbf{x}_{k+1} - \mathbf{x}_k) \cdot \gamma_{AB} \tilde{S}_k^B] \quad (5.1)$$

Recalling the “shortcut” construction of the preceding section, we see that we can modify these supercharges by conjugating $Q_1 + iQ_2$ with a nonunitary similarity transformation of the form

$$V = \prod_k v((\mathbf{x}_{k+1} - \mathbf{x}_k)^2, \tilde{S}_k) \quad (5.2)$$

which will change the harmonic nearest neighbor potential to an essentially arbitrary one, and at the same time introduce terms into Q_A cubic in S, \tilde{S} . This construction then defines a supersymmetric chain dynamics with short-range forces. The one particle quantum mechanics discussed in this paper can be used in the approximate calculational scheme developed in [2]. The preservation of supersymmetry in this approximation scheme will automatically enforce the various subtractions necessary in passing to the continuum limit.

The reader may well wonder why we are so concerned to have short range bonding forces. After all, the continuum limit of the string-bit polymers should wash out the details of the bonding force, so why not be content with a harmonic one? This would probably be a satisfactory position if we were only interested in understanding free superstring and its *perturbative* interactions. But our real aim is a bit higher, namely to provide a physically sound basis for superstring theory using string-bits as building blocks. In other words, we want the dynamics of string-bits themselves to be physically sensible. From a non-perturbative point of view a bit in one piece of superstring can interact directly with one in another piece of superstring: a model of polymer bound states based solely on nearest neighbor interactions is an approximate description, albeit one that can be singled out by, for example, a $1/N_c$ expansion. A harmonic force between nearest neighbor bits in a polymer would also, in higher approximations, be present between non-neighbor bits, including bits on different polymers. This would strongly violate one of the most fundamental physical properties of our world, cluster decomposition. While it is barely conceivable that delicate cancellations could be arranged to skirt this disaster, we think a much more satisfactory and robust resolution of the difficulty is to forbid the presence of such forces in the fundamental dynamics from the beginning. But then we are faced with the problems struggled with in this paper.

What can be said about our inability to extend our construction to $d = 8$, the critical dimension for superstring? Perhaps our range of search was too narrow. To broaden it we would have to allow the momentum to enter the supercharges in a more complicated way. Unfortunately, this greatly increases the technical complications in enforcing the superalgebra. Also one would have to guard against merely reproducing, after much labor, the trivial representation of the superalgebra described in the introduction. Using the methods of this paper, however, we can easily set up a dynamics with the degrees of freedom necessary for critical superstring, but with only that part of supersymmetry associated with the $d = 2$ subspace realized. This might be completely satisfactory. After all, our physical world exists in 4 dimensional space-time, which corresponds to this $d = 2$ subspace. The practical virtues of supersymmetry, namely an energy spectrum bounded from below and the enforcement of necessary cancellations, will be retained with such a partial realization of supersymmetry. Finally, any supersymmetry that remains has to be broken to account for the absence of supersymmetry in our world. Perhaps it is not such a tragedy if most of the supersymmetry of perturbative superstring were simply not present in the underlying dynamics, but is rather an artifact of perturbation theory.

Let’s spell out this possibility in a little more detail. To focus on the supersymmetry we wish to preserve, cast the $SO(8)$ superalgebra in the language of $U(1) \times SU(4)$ where the $U(1)$ factor describes rotations in the $d = 2$ plane, and $SU(4) \simeq SO(6)$ describes rotations in the remaining 6 directions (see, for example, Chapter 11 of [10]). Then the 8 hermitian supercharges $Q^{\dot{a}}$ are replaced by four non-hermitian charges Q^A , where A labels the components of a **4** representation of $SU(4)$.

Then $Q_A \equiv Q^{A\dagger}$ transforms as a $\bar{4}$. Then the $SO(8)$ superalgebra takes on the appearance of an $N = 4$ extended $SO(2)$ superalgebra:

$$\{Q^A, Q^B\} = 0 \quad \{Q^A, Q_B\} = \delta_B^A H. \quad (5.3)$$

With harmonic nearest neighbor forces this algebra is fully satisfied without difficulty. Modifying these forces to short range via the device of a nonunitary similarity transformation will only preserve the first of Eq. (5.3). But breaking the $SU(4)$ internal symmetry by singling out one direction, say $Q \equiv Q^1$, allows the specification of a dynamics

$$H \equiv \{Q, Q^\dagger\} \quad (5.4)$$

consistent with an $N = 1$ $SO(2)$ supersymmetry. This would be the only vestige of supersymmetry present in the underlying dynamics.

Finally we must note that our task of incorporating $d = 2$ supersymmetry into string-bit models is still far from completion. We have dealt with the dynamics of a “bare polymer” chain with only nearest neighbor interactions. This approximation arises in an $N_c \rightarrow \infty$ limit of a second-quantized description, in which the fields are $N_c \times N_c$ matrices, as discussed in [3]. We have not dealt with the problem of extending the supersymmetry we have developed for the first-quantized chain Hamiltonian to the second quantized Hamiltonian. This is a non-trivial task we have yet to tackle.

A The Case $d = 4$

In this appendix we analyze the case of four dimensions which was not included in the main discussion because the supercharges must be nonhermitian, entailing several differences in detail. In four dimensions the Dirac matrices can be chosen to be

$$\gamma = \begin{pmatrix} \mathbf{0} & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (A.1)$$

where we use bold-face to denote the first three components of a four-vector and plain type-face to denote all four components: $x = (\mathbf{x}, x^4)$. In this representation the chirality matrix is diagonal. We anticipate that the Weyl restriction is necessary for harmonic forces, so we let our spinor variables be two component from the beginning, and consider the following forms for supercharges:

$$\begin{aligned} Q_{\dot{a}} &= (p^4 - i\boldsymbol{\sigma} \cdot \mathbf{p})_{\dot{a}c} S^c + k(x^4 - i\boldsymbol{\sigma} \cdot \mathbf{x})_{\dot{a}c} \tilde{S}^c \\ Q_{\dot{a}}^\dagger &= (p^4 + i\boldsymbol{\sigma} \cdot \mathbf{p})_{c\dot{a}} S^{c\dagger} + k(x^4 + i\boldsymbol{\sigma} \cdot \mathbf{x})_{c\dot{a}} \tilde{S}^{c\dagger} \end{aligned} \quad (A.2)$$

The required superalgebra is then

$$\{Q_{\dot{a}}, Q_{\dot{b}}\} = 0 \quad \{Q_{\dot{a}}, Q_{\dot{b}}^\dagger\} = 4\delta_{\dot{a}\dot{b}} H. \quad (A.3)$$

With harmonic forces, the first of Eq. (A.3) follows as a consequence of the following identity satisfied by Pauli matrices:

$$\boldsymbol{\sigma}_{\dot{a}c} \cdot \boldsymbol{\sigma}_{\dot{b}d} - \delta_{\dot{a}c} \delta_{\dot{b}d} = 2\sigma_{\dot{a}\dot{b}}^2 \sigma_{cd}^2 \quad (A.4)$$

and the fact that σ^2 is antisymmetric. The second of Eq. (A.3) requires a variant of this identity, easily derived by using $\boldsymbol{\sigma}^T = -\sigma^2 \boldsymbol{\sigma} \sigma^2$:

$$\boldsymbol{\sigma}_{\dot{a}c} \cdot \boldsymbol{\sigma}_{\dot{d}\dot{b}} + \delta_{\dot{a}c} \delta_{\dot{d}\dot{b}} = 2\delta_{\dot{a}\dot{b}} \delta_{cd}. \quad (A.5)$$

The resulting supersymmetric oscillator Hamiltonian is

$$H = \frac{1}{2}(p \cdot p + k^2 x \cdot x) - \frac{ik}{2}(S^c \tilde{S}^{c\dagger} - \tilde{S}^c S^{c\dagger}), \quad (\text{A.6})$$

where $v \cdot v = \mathbf{v}^2 + (v^4)^2$ for a four-vector v .

Now we want to explore the possibility of replacing the harmonic force by a short-range one. In line with our main discussion we impose the *ansatz* that the momentum dependence of the supercharges is untouched

$$Q_{\dot{a}} = (p^4 - i\boldsymbol{\sigma} \cdot \mathbf{p})_{\dot{a}c} S^c + \hat{Q}_{\dot{a}}(x, \tilde{S}, \tilde{S}^\dagger, S, S^\dagger). \quad (\text{A.7})$$

Just as in the main text, look first at the terms in the anticommutators linear in p . The first equation of Eq. (A.3) implies

$$\begin{aligned} \{S^{\dot{a}}, \hat{Q}_{\dot{b}}\} + \{S^{\dot{b}}, \hat{Q}_{\dot{a}}\} &= 0 \\ \sigma_{\dot{a}c}^k \{S^c, \hat{Q}_{\dot{b}}\} + \sigma_{bc}^k \{S^c, \hat{Q}_{\dot{a}}\} &= 0. \end{aligned} \quad (\text{A.8})$$

Putting $\dot{b} = \dot{a}$, multiplying the second equation by $\sigma_{\dot{a}a}^k$, summing over k , and using Eq. (A.5) and the first equation, leads to $\{S^c, \hat{Q}_{\dot{a}}\} = 0$, *i.e.* $\hat{Q}_{\dot{a}}$ is independent of S^\dagger . Next the second equation of Eq. (A.3) implies

$$\begin{aligned} \{S^{\dot{a}}, \hat{Q}_{\dot{b}}^\dagger\} + \{S^{\dot{b}\dagger}, \hat{Q}_{\dot{a}}\} &= \delta_{\dot{a}\dot{b}} V^4 \\ -i\sigma_{\dot{a}c}^k \{S^c, \hat{Q}_{\dot{b}}^\dagger\} + i\sigma_{cb}^k \{S^{c\dagger}, \hat{Q}_{\dot{a}}\} &= \delta_{\dot{a}\dot{b}} V^k. \end{aligned} \quad (\text{A.9})$$

Putting $\dot{b} = \dot{a}$, multiplying the second equation by $\sigma_{\dot{a}a}^k$, summing over k , and using Eq. (A.4), Eq. (A.5) and the first equation, allows the determination of $\{S^{c\dagger}, \hat{Q}_{\dot{a}}\}$ in terms of V . Repeating these manipulations, multiplying instead by $\sigma_{\dot{a}a}^k$ determines $\{S^c, \hat{Q}_{\dot{a}}^\dagger\}$ in terms of V . The results are

$$\begin{aligned} \{S^{c\dagger}, \hat{Q}_{\dot{a}}\} &= \frac{1}{2}\delta_{\dot{a}c} V^4 - \frac{i}{2}\boldsymbol{\sigma}_{\dot{a}c} \cdot \mathbf{V} \\ \{S^c, \hat{Q}_{\dot{a}}^\dagger\} &= \frac{1}{2}\delta_{\dot{a}c} V^4 + \frac{i}{2}\boldsymbol{\sigma}_{c\dot{a}} \cdot \mathbf{V}. \end{aligned} \quad (\text{A.10})$$

Since the left hand sides of these two equations are hermitian conjugates of one another, it follows that V^4, \mathbf{V} are hermitian. Also the l.h.s side of the first is independent of S^\dagger and that of the second is independent of S , it follows that V^4, \mathbf{V} are independent of both S and S^\dagger , and that $\hat{Q}_{\dot{a}}$ is at most linear in S . In summary, the p dependent terms of the superalgebra imply that $Q_{\dot{a}}$ has the form

$$\begin{aligned} Q_{\dot{a}} &= (p^4 + A^4(x, \tilde{S}, \tilde{S}^\dagger) - i\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}(x, \tilde{S}, \tilde{S}^\dagger)))_{\dot{a}c} S^c \\ &\quad + M_{\dot{a}}(x, \tilde{S}, \tilde{S}^\dagger), \end{aligned} \quad (\text{A.11})$$

a result entirely analogous to Eq. (3.10).

Next we look at the momentum independent terms bilinear in S, S^\dagger . Isolating the bilinear terms in $\{Q_{\dot{a}}, Q_{\dot{b}}^\dagger\}$ produces

$$- \left\{ i(\delta_{\dot{a}c}\sigma_{\dot{d}b}^k + \sigma_{\dot{a}c}^k\delta_{\dot{d}b})[D^{\dot{d}}, D^{\dot{b}}] + \sigma_{\dot{a}c}^k\sigma_{\dot{d}b}^l[D^{\dot{d}}, D^{\dot{b}}] \right\} S^{d\dagger} S^c, \quad (\text{A.12})$$

where we have abbreviated $D \equiv p + A$. We require two Fierz style identities

$$\begin{aligned}\sigma_{\dot{a}c}^k \sigma_{db}^l - \sigma_{\dot{a}c}^l \sigma_{db}^k &= \delta_{\dot{a}b} i \epsilon^{lkm} \sigma_{dc}^m + i \epsilon^{lmk} \sigma_{\dot{a}b}^m \delta_{dc} \\ \delta_{\dot{a}c} \sigma_{db}^k + \sigma_{\dot{a}c}^k \delta_{db} &= \delta_{\dot{a}b} \sigma_{dc}^k + \sigma_{\dot{a}b}^k \delta_{dc}.\end{aligned}\tag{A.13}$$

The terms proportional to $\delta_{\dot{a}b}$ are compatible with the superalgebra, leaving the ones which must vanish:

$$-i \sigma_{\dot{a}b}^m \delta_{dc} ([D^4, D^m] + \frac{1}{2} \epsilon^{lmk} [D^k, D^l]) = 0.\tag{A.14}$$

Since the three Pauli matrices are independent, the following three constraints must hold

$$[D^4, D^m] + \frac{1}{2} \epsilon^{lmk} [D^k, D^l] = 0,\tag{A.15}$$

which is just the statement that the antisymmetric “field strength” $F^{\mu\nu}$ is self dual. The analysis of the bilinear terms in $\{Q_{\dot{a}}, Q_{\dot{b}}\} = 0$ turns out to add no new constraints. The easiest way to see this is to use in place of $Q_{\dot{b}}$ the related operators

$$\begin{aligned}(\sigma^2 Q)_{\dot{b}} &= (p^4 + A^4(x, \tilde{S}, \tilde{S}^\dagger) \\ &\quad + i \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}(x, \tilde{S}, \tilde{S}^\dagger))_{\dot{a}b} (\sigma^2 S)^d \\ &\quad + (\sigma^2 M)_{\dot{b}}(x, \tilde{S}, \tilde{S}^\dagger),\end{aligned}\tag{A.16}$$

so the bilinear terms in $\{Q_{\dot{a}}, (\sigma^2 Q)_{\dot{b}}\}$ are identical to those in $\{Q_{\dot{a}}, Q_{\dot{b}}^\dagger\}$ with $S^{d\dagger} S^c$ replaced by $(\sigma^2 S)^d S^c$. The one difference is that the terms proportional to $\delta_{\dot{a}b}$ must now also vanish. But they do because those terms involve the factor $\sigma_{dc}^m (\sigma^2 S)^d S^c = -(\sigma^2 \sigma^m)_{dc} S^d S^c$ which vanishes identically because $\sigma^2 \sigma^m$ are symmetric matrices. Thus we find that the case $d = 4$ is sort of intermediate between $d = 2$ and $d = 8$: some but not all linear combinations of the components of F must vanish: F must be self-dual.

Assuming self-duality the anticommutators of supercharges reduce to

$$\begin{aligned}\{Q_{\dot{a}}, (\sigma^2 Q)_{\dot{b}}\} &= [M_{\dot{a}}, (D^4 + i \boldsymbol{\sigma} \cdot \mathbf{D})_{\dot{a}b}] (\sigma^2 S)^d + [(\sigma^2 M)_{\dot{b}}, (D^4 - i \boldsymbol{\sigma} \cdot \mathbf{D})_{\dot{a}c}] S^c + \{M_{\dot{a}}, (\sigma^2 M)_{\dot{b}}\} = 0 \\ \{Q_{\dot{a}}, Q_{\dot{b}}^\dagger\} &= ((D^4)^2 + \mathbf{D}^2) \delta_{\dot{a}b} + [M_{\dot{a}}, (D^4 + i \boldsymbol{\sigma} \cdot \mathbf{D})_{\dot{a}b}] S^{d\dagger} \\ &\quad + [M_{\dot{b}}^\dagger, (D^4 - i \boldsymbol{\sigma} \cdot \mathbf{D})_{\dot{a}c}] S^c + \{M_{\dot{a}}, M_{\dot{b}}^\dagger\} = 4 \delta_{\dot{a}b} H.\end{aligned}\tag{A.17}$$

The linear terms in S, S^\dagger must separately be proportional to $\delta_{\dot{a}b}$ in the second equation and must vanish in the first equation. From the second equation we conclude that

$$[M_{\dot{a}}, (D^4 + i \boldsymbol{\sigma} \cdot \mathbf{D})_{\dot{a}b}] = \Psi_d \delta_{\dot{a}b},\tag{A.18}$$

which implies also its hermitian conjugate. Once this holds the corresponding linear terms in the first equation automatically vanish, as can be easily shown. Finally the terms independent of S imply

$$\{M_{\dot{a}}, M_{\dot{b}}^\dagger\} = 4V \delta_{\dot{a}b} \quad \{M_{\dot{a}}, M_{\dot{b}}\} = 0.\tag{A.19}$$

In summary, supersymmetric quantum mechanics in $d = 4$ will be realized if solutions to Eq. (A.15), Eq. (A.18), and Eq. (A.19) can be found. In these equations M and A are allowed to depend only on the coordinates \mathbf{x} and on the fermionic variables $\tilde{S}, \tilde{S}^\dagger$. We defer the search for such solutions and a study of their properties for a later time.

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